Application of Runge-Kutta to Solving the Nonlinear Ordinary Differential Equation of a Deformed Cylindrical Compressible Blatz-Ko Material Under Torsional Effects.

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Abstract
This paper discussed the homogeneous, isotropic, compressible nonlinearly elastic cylindrical Blatz-Ko material deforming under pure torsion. The mathematical model of the radial deformation of the structure resulted into a highly nonlinear second-order ordinary differential equation with boundary conditions and the solution was obtained using Runge-Kutta fourth order method and implemented using MATLAB software. The results show that deformations decrease towards the origin and a decrease in volume is observed. The effects of stress and applied pressure on the material under study were compared and results showed that the stress increased as the radius of the material decreases.

Keywords: *Blatz-Ko, Matlab, Runge-Kutta, Stress, Deformation, Hyperelastic, Material, mechanics, Isotropic, and compressible.*

INTRODUCTION

Understanding how materials deform under different loads is important for engineering and design in materials science and continuum mechanics. Hyperelastic materials, which can withstand massive deformations and return to their original shape, play an important role in biomechanics, soft robotics, and materials engineering. Unlike linear elasticity models, finite deformation theories are required to represent the complicated behavior of these materials.

Torsional stresses frequently cause considerable deformation in cylindrical structures such as blood veins and elastomeric tubes. Although incompressible hyperelastic materials, such as rubbers, have well-known solutions for finite torsion (Rivlin, 1997), compressible materials provide new issues. The lack of isochoric restrictions complicates their deformation behavior, necessitating particular strain-energy functions for correct modeling (Polignone and Horgan, 1991; Kirkinis and Ogden, 2002).

Compressive materials under torsion have been studied theoretically and empirically to gain a better understanding of their behavior. Shrivastava et al. (1982) and Valiollahi et al. (2019) contributed insights into the torque required for sustaining deformation, whereas Currie and Hayes (1981) offered constitutive relations for pure torsion. This study will build on these foundations by investigating the torsional response of compressible Blatz-Ko materials using the Runge-Kutta method for numerical solutions.

Chibueze and Julius (2021) centred on isotropic, incompressible hollow cylindrical structure that is deforming under pure azimuthal shear. Their focus was on development of the constitutive law

for an initially stressed material that has no intrinsic material symmetry. Kassianidis *et al.*, (2008) studied the problem of azimuthal shear of a cylinder subject to finite deformation, a general form of strain energy function was used and a closed form solution was obtained for a reinforced Neo Hookean material which was used to determine the domain of strong ellipticity in terms of the relationship between the shear strain and the angle.
Bechir *et al.,* (2006) studied the behaviour of isotropic and incompressible vulcanized natural rubbers and that of quasi-incompressible carbon black filled vulcanized natural rubbers considering both theoretically and experimentally obtained solutions by generalizing the neo-Hookean model and derived an original form of the strain energy density function. Jiang and Ogden (2000) provided some new solutions for the axial shear of a circular cylindrical tube of compressible isotopic elastic material and discussed explicit solution for several forms of the strain energy function, analyzing the plain strain characterized of finite torsion shear cylinder of a compressible elastic material.
This work intends to bridge this gap by numerically analyzing the torsional behavior of compressible Blatz-Ko materials using the Runge-Kutta technique, revealing new insights into their mechanical response and broadening the scope of finite deformation theories for compressible materials.

MATHEMATICAL FORMULATION

Consider the cylindrical deformation of a cylinder where the deformation takes the point with the cylindrical coordinates (r,θ,z) in the reference configuration to the coordinates (R,ϕ,Z). Let e_R represent a unit vector normal to the cylinder e_ϕ represent a unit vector circumferential to the cylinder chosen to make $\{e_R, e_\phi, e_Z\}$ a right-handed triad. e_Z is parallel to the k vector.
The triad of vectors $\{e_R, e_\phi, e_Z\}$ is an orthonormal basis. Consequently, tensors can be represented as components in this basis. In particular, a general second order tensor **S** can be represented as a 3x3 matrix given as

$$\bar{S} = \begin{bmatrix} S_{RR} & S_{R\phi} & S_{RZ} \\ S_{\phi R} & S_{\phi\phi} & S_{\phi Z} \\ S_{ZR} & S_{Z\phi} & S_{ZZ} \end{bmatrix} \tag{1}$$

The Deformation Gradient Tensor: Let us define the gradient operator, which, in cylindrical-polar coordinates, has the representation

$$\nabla = e_R \frac{\partial}{\partial r} + \frac{e_\phi}{R}\frac{\partial}{\partial \theta} + e_Z \frac{\partial}{\partial z} \tag{2}$$

In addition, the nonzero derivatives of the basis vectors are

$$\nabla = \frac{\partial e_R}{\partial r} = e_\phi \qquad \frac{\partial e_\phi}{\partial \theta} = -e_R \tag{3}$$

Then the deformation gradient tensor F can be represented as a dyadic product of the tensor S with the gradient operator as

$$\bar{F} = \bar{S} \otimes \nabla = \begin{bmatrix} \frac{\partial S_R}{\partial r} & \frac{1}{R}\frac{\partial S_R}{\partial \theta} & \frac{\partial S_R}{\partial z} \\ S_R \frac{\partial S_\phi}{\partial r} & \frac{R}{r}\frac{\partial S_\phi}{\partial \theta} & S_R \frac{\partial S_\phi}{\partial z} \\ \frac{\partial S_Z}{\partial r} & \frac{1}{r}\frac{\partial S_Z}{\partial \theta} & \frac{\partial S_Z}{\partial z} \end{bmatrix} \tag{4}$$

Deformation Measures: Let **B** represent the Left Cauchy-Green deformation tensor, then,
$\bar{B} = \bar{F}.\bar{F}^T \Rightarrow \bar{B} = F_{ik}F_{jk}$; $J = \det(\bar{F})$

The invariants of **B** are given as:

$$I_1 = trace\,\bar{B} = \sum_{k=1}^{3} B_{kk} \tag{5}$$

$$I_2 = \frac{1}{2}\left(I_1^{\,2} - trace\,\bar{B}^2\right) \tag{6}$$

$$I_3 = \det(\bar{B}) = J^2 \tag{7}$$

Let e_1, e_2 and e_3 denote the three eigenvalues of **B**. The principal stretches are given as
$\lambda_1 = \sqrt{e_1}, \quad \lambda_2 = \sqrt{e_2}, \quad \lambda_3 = \sqrt{e_3}$

Stress Measures

Usually stress-strain laws are given as equations relating Cauchy stress, (`true' stress) σ_{ij} to left Cauchy-Green deformation tensor. The Cauchy ("true") stress represents the force per unit deformed area in the solid and is defined by

$$n_i \sigma_{ij} = \lim_{dA \to 0} \frac{dP_j^{(n)}}{dA} \tag{8}$$

Note that, by definition, if the solid is subjected to some history of strain, the rate of change of the strain energy density $W(\mathbf{F})$ must equal the rate of mechanical work done on the material per unit reference volume. Equivalently, W only depends on **F** through its principal stretches $\lambda_1, \lambda_2, \lambda_3$ (the square roots of the principal values of **B**).

The ratio of the deformed configuration to the undeformed configuration is called a stretch, i.e.,

Stretch(λ_1) $= \frac{|dR|}{|dr|}$, Stretch in the radial direction

Stretch(λ_2) $= \frac{|d\phi|}{|d\theta|}$, Stretch in the angular direction

Stretch(λ_3) $= \frac{|dZ|}{|dz|}$, Stretch in the azimuthal direction

With a slight abuse of notation, we write $W = W(I_1, I_2, I_3)$.

To compute the explicit form of the Cauchy stress tensor for a compressible material in terms of the invariants and their derivatives we use

$$\frac{\partial I_1}{\partial \bar{F}} = 2\bar{F}^T, \quad \frac{\partial I_2}{\partial \bar{F}} = 2I_1\bar{F}^T - 2\bar{F}^T\bar{B}, \quad \frac{\partial I_3}{\partial \bar{F}} = 2I_3\bar{F}^{-1} \text{ and } \bar{\sigma} = \omega_0 I + \omega_1\bar{B} + \omega_2\bar{B}^2,$$

Where the constants ω_i depend on the invariants and are given explicitly by

$$\omega_0 = 2J\frac{\partial W}{\partial I_3} - p, \quad \omega_1 = 2J^{-1}\frac{\partial W}{\partial I_1} + 2J^{-1}\frac{\partial W}{\partial I_2}I_1, \quad \omega_2 = -2J^{-1}\frac{\partial W}{\partial I_2}.$$

where p is the hydrostatic pressure and we choose p = 0 for compressible materials.

Development of Model Equation for the Finite Deformation of a Cylindrical Compressible Hyperelastic Material

Let us consider the torsional finite deformation of an elastic solid circular cylinder of radius a due to applied twisting moments at its ends,

$$R = \eta R(r) \qquad 0 < r \le a$$

$$\phi = \theta + \tau z \qquad 0 < \theta \le 2\pi$$

$$Z = z \qquad 0 < z \le L$$

where (r,θ,z) and (R,ϕ,Z) are the cylindrical coordinates in the reference and in the current configurations, respectively, $dR/dr < 0,$ and the constant $\tau > 0$ is the twist per unit undeformed length. η is a positive constant which accounts for initial compressibility. Let us consider the strain energy function in terms of the first three principal invariants of **B**, $W = W(I_1, I_2, I_3)$. The deformation gradient tensor **F** is given by

$$\bar{F} = \begin{bmatrix} \eta R' & 0 & 0 \\ 0 & \frac{R}{r} & \tau R \\ 0 & 0 & 1 \end{bmatrix} \tag{9}$$

and the physical components of **B** and **B**^{2} are given by

$$\bar{B} = \bar{F}.\bar{F}^T = \begin{bmatrix} \eta^2 R'^2 & 0 & 0 \\ 0 & \frac{R^2}{r^2} + \tau^2 R^2 & \tau R \\ 0 & \tau R & 1 \end{bmatrix}, \quad \bar{B}^2 = \begin{bmatrix} \eta^4 R'^4 & 0 & 0 \\ 0 & \left(\frac{R^2}{r^2} + \tau^2 R^2\right)^2 + \tau^2 R^2 & \tau R\left(\frac{R^2}{r^2} + \tau^2 R^2\right) + \tau R \\ 0 & \tau R\left(\frac{R^2}{r^2} + \tau^2 R^2\right) + \tau R & \tau^2 R^2 + 1 \end{bmatrix} \tag{10}$$

The first three principal strain invariants are

$$I_1 = \eta^2 R'^2 + \frac{R^2}{r^2} + \tau^2 R^2 + 1 \tag{11}$$

$$I_2 = \frac{1}{2}\left(\left(1+\eta^2 R'^2 + \frac{R^2}{r^2} + \tau^2 R^2\right)^2 - \left(\eta^4 R'^4 + \left(\frac{R^2}{r^2} + \tau^2 R^2\right)^2 + 2\tau^2 R^2 + 1\right)\right) \tag{12}$$

$$I_2 = \frac{\eta^2 R^2 R'^2}{r^2} + \frac{R^2}{r^2} + \eta^2 \tau^2 R^2 R'^2 + \eta^2 R'^2 \tag{13}$$

$$I_3 = \frac{\eta^2 R^2 R'^2}{r^2} \tag{14}$$

We obtain the physical components of the Cauchy stress as

$$\bar{\sigma} = \begin{bmatrix} \sigma_{RR} & \sigma_{R\phi} & \sigma_{RZ} \\ \sigma_{\phi R} & \sigma_{\phi\phi} & \sigma_{\phi Z} \\ \sigma_{ZR} & \sigma_{Z\phi} & \sigma_{ZZ} \end{bmatrix} = \omega_0 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \omega_1 \begin{bmatrix} \eta^2 R'^2 & 0 & 0 \\ 0 & \frac{R^2}{r^2} + \tau^2 R^2 & \tau R \\ 0 & \tau R & 1 \end{bmatrix}$$

$$+\omega_2 \begin{bmatrix} \eta^4 R'^4 & 0 & 0 \\ 0 & \left(\frac{R^2}{r^2} + \tau^2 R^2\right)^2 + \tau^2 R^2 & \tau R\left(\frac{R^2}{r^2} + \tau^2 R^2\right) + \tau R \\ 0 & \tau R\left(\frac{R^2}{r^2} + \tau^2 R^2\right) + \tau R & \tau^2 R^2 + 1 \end{bmatrix} \tag{15}$$

We shall consider the Blatz-Ko strain energy function for compressible nonlinear elastic behaviour of the material under study given by Blatz and Ko (1962)

$$W = \frac{\mu}{2}\left(\frac{I_2}{I_3} + 2I_3^{1/2} - 5\right) \tag{16}$$

Where: μ is the initial shear modulus of the material.

$$\frac{\partial W}{\partial I_2} = \frac{\mu}{2I_3} = \frac{\mu r^2}{2\eta^2 R^2 R'^2}, \quad \frac{\partial W}{\partial I_3} = \frac{\mu}{2}\left(\frac{r}{\eta R R'} - \frac{r^2}{\eta^2 R^2 R'^2} - \frac{r^2}{\eta^4 R^2 R'^4} - \frac{r^4 \tau^2}{\eta^2 R^2 R'^2} - \frac{r^4}{\eta^2 R^4 R'^2}\right) \tag{17}$$

$$\omega_0 = 2J\frac{\partial W}{\partial I_3} = \frac{2\eta R R'}{r}\left(\frac{\mu}{2}\left(\frac{r}{\eta R R'} - \frac{r^2}{\eta^2 R^2 R'^2} - \frac{r^2}{\eta^4 R^2 R'^4} - \frac{r^4 \tau^2}{\eta^2 R^2 R'^2} - \frac{r^4}{\eta^2 R^4 R'^2}\right)\right) \tag{18}$$

$$\omega_0 = 2J\frac{\partial W}{\partial I_3} = \mu\left(1 - \frac{r}{\eta R R'} - \frac{r}{\eta^3 R R'^3} - \frac{r^3 \tau^2}{\eta R R'} - \frac{r^3}{\eta R^3 R'}\right) \tag{19}$$

$$\omega_1 = 2J^{-1}\frac{\partial W}{\partial I_2} I_1 = \frac{\mu r^3}{\eta R^3 R'} + \frac{\mu r}{\eta^3 R R'^3} + \frac{\tau^2 \mu r^3}{\eta^3 R R'^3} + \frac{\mu r^3}{\eta^3 R^3 R'^3} \tag{20}$$

$$\omega_2 = -2J^{-1}\frac{\partial W}{\partial I_2} = -\frac{\mu r^3}{\eta^3 R^3 R'^3}. \tag{21}$$

Then,

$$\sigma_{RR}=\mu-\frac{\mu r}{\eta^{3}RR'^{3}},\quad \sigma_{\phi\phi}=\mu-\frac{\mu r^{3}}{\eta R^{3}R'} \tag{22}$$

The equilibrium equations in cylindrical coordinates is given as

$$\frac{d\sigma_{RR}}{dr}+\frac{1}{R}\frac{d\sigma_{R\phi}}{d\theta}+\frac{d\sigma_{RZ}}{dz}+\frac{1}{R}\left(\sigma_{RR}-\sigma_{\phi\phi}\right)+\rho b_{R}=\rho a_{R} \tag{23}$$

$$\frac{d\sigma_{\phi R}}{dr}+\frac{1}{R}\frac{d\sigma_{\phi\phi}}{d\theta}+\frac{d\sigma_{\phi Z}}{dz}+\frac{2}{R}\sigma_{R\phi}+\rho b_{\phi}=\rho a_{\phi} \tag{24}$$

$$\frac{d\sigma_{ZR}}{dr}+\frac{1}{R}\frac{d\sigma_{Z\phi}}{d\theta}+\frac{d\sigma_{ZZ}}{dz}+\frac{1}{R}\sigma_{ZR}+\rho b_{Z}=\rho a_{Z} \tag{25}$$

where

b_R, b_ϕ and b_Z are the components of the body force

a_R, a_ϕ and b_Z are the components of acceleration.

The equilibrium equations in the absence of any body force are reduced to

$$\frac{d\sigma_{RR}}{dr}+\frac{1}{R}\frac{d\sigma_{R\phi}}{d\theta}+\frac{d\sigma_{RZ}}{dz}+\frac{1}{R}\left(\sigma_{RR}-\sigma_{\phi\phi}\right)=0 \tag{26}$$

$$\frac{d\sigma_{\phi R}}{dr}+\frac{1}{R}\frac{d\sigma_{\phi\phi}}{d\theta}+\frac{d\sigma_{\phi Z}}{dz}+\frac{2}{R}\sigma_{R\phi}=0 \tag{27}$$

$$\frac{d\sigma_{ZR}}{dr}+\frac{1}{R}\frac{d\sigma_{Z\phi}}{d\theta}+\frac{d\sigma_{ZZ}}{dz}+\frac{1}{R}\sigma_{ZR}=0 \tag{28}$$

We substitute the obtained component stresses into the equilibrium equations above, we get,

$$\frac{d}{dr}\left(\mu-\frac{\mu r}{\eta^{3}RR'^{3}}\right)+\frac{1}{R}\left(-\frac{\mu r}{\eta^{3}RR'^{3}}+\frac{\mu r^{3}}{\eta R^{3}R'}\right)=0 \tag{29}$$

$$-\left[\frac{\mu RR'-\mu r(3RR''+R'^{2})}{\eta^{3}R^{2}R'^{4}}\right]+\frac{\mu\eta^{2}r^{3}R'^{2}-\mu rR^{2}}{\eta^{3}R^{4}R'^{3}}=0 \tag{30}$$

Simplifying further,

$$3\mu rR^{3}R''+\mu\eta^{2}r^{3}R'^{3}+\mu rR^{2}R'^{2}-\left(\mu R^{3}+\mu rR^{2}\right)R'=0 \tag{31}$$

The differential equation obtained above is highly nonlinear and no closed form solution exists hence we will apply numerical method specifically the RK4 method to solve it.

Let the quantity dR/dr mathematically describes how the material has been deformed from the reference configuration to the current configuration. Then we can say that

$$\left.\frac{dR}{dr}\right|_{r=1}=-p \tag{32}$$

where p is the initial pressure applied to the material externally. The negative sign does not necessarily mean the change is negative but rather it means a contraction or decrease in volume.

Equation (32) is physically true since an initially applied pressure will determine the magnitude of the quantity dR/dr. For the purpose of this research we take $p = 0.25\,N/m^2$. To this end, we choose our initial condition as $R(1) = 5,\ R'(1) = -0.25$ (33)

We split the equation (31) into a system of first order ordinary differential equations by letting

$$R'(r) = u(r) \Rightarrow R''(r) = u'(r) \tag{33}$$

We substitute equation (33) into equation (31), we get the system

$$\left.\begin{aligned} &\frac{dR}{dr} = u \\ &\frac{du}{dr} = \frac{(R+r)}{3rR} - \frac{1}{3R}u^2 - \frac{\eta^2 r^2}{3R^2}u^3 \\ &R(1) = 5, \quad u(1) = -0.25 \end{aligned}\right\} \tag{34}$$

We solve the system (34) by Runge-Kutta fourth order method to obtain the radial coordinates in the current configuration. The stresses on the material can be computed from the obtained R values using the formula

$$\sigma = \frac{|R-r|}{r}E \tag{35}$$

Where E is the elastic modulus. We take E = 0.09 (Patel, 2008).

We choose the step size $h = 0.1$ for $1 \le r \le 2$. Hence. $r_0 = 1,\ R_0 = 5,\ \eta = 0.5$ and $u_0 = -0.25.$

$$\begin{aligned} &f(r,R,u) = u \\ &g(r,R,u) = \frac{(R+r)}{3rR} - \frac{1}{3R}u^2 - \frac{\eta^2 r^2}{3R^2}u^3 \end{aligned} \tag{36}$$

First iteration

$k_1 = hf(r_0, R_0, u_0) = -0.025$, $l_1 = hg(r_0, R_0, u_0) = 0.0395841667$

$k_2 = hf\left(r_0 + \frac{h}{2}, R_0 + \frac{k_1}{2}, u_0 + \frac{l_1}{2}\right) = -0.023020791$, $l_2 = hg\left(r_0 + \frac{h}{2}, R_0 + \frac{k_1}{2}, u_0 + \frac{l_1}{2}\right) = 0.03807593778$

$k_3 = hf\left(r_0 + \frac{h}{2}, R_0 + \frac{k_2}{2}, u_0 + \frac{l_2}{2}\right) = -0.0230962031$, $l_3 = hg\left(r_0 + \frac{h}{2}, R_0 + \frac{k_2}{2}, u_0 + \frac{l_2}{2}\right) = 0.03807236517$

$k_4 = hf(r_0 + h, R_0 + k_3, u_0 + l_3) = -0.021192763,\ l_4 = hg(r_0 + h, R_0 + k_3, u_0 + l_3) = 0.036700443$

$$R_1 = R_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 4.976928874$$

Second iteration

$k_1 = hf(r_1, R_1, u_1) = -0.021190313$ $l_1 = hg(r_1, R_1, u_1) = 0.0367004802$

$k_2 = hf\left(r_1 + \frac{h}{2}, R_1 + \frac{k_1}{2}, u_1 + \frac{l_1}{2}\right) = -0.019355289,\ l_2 = hg\left(r_1 + \frac{h}{2}, R_1 + \frac{k_1}{2}, u_1 + \frac{l_1}{2}\right) = 0.03544644046$

$$k_3 = hf\left(r_1 + \frac{h}{2}, R_1 + \frac{k_2}{2}, u_1 + \frac{l_2}{2}\right) = -0.019417991049,\ l_3 = hg\left(r_1 + \frac{h}{2}, R_1 + \frac{k_2}{2}, u_1 + \frac{l_2}{2}\right) = 0.035443620545$$

$$k_4 = hf\left(r_1 + h, R_1 + k_3, u_1 + l_3\right) = -0.0176459510176,\ l_4 = hg\left(r_1 + h, R_1 + k_3, u_1 + l_3\right) = 0.03429264585$$

$$R_2 = R_1 + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right) = 4.9575317371$$

Third iteration

$$k_1 = hf(r_2, R_2, u_2) = -0.017644092,\ l_1 = hg(r_2, R_2, u_2) = 0.03429266241$$

$$k_2 = hf\left(r_2 + \frac{h}{2}, R_2 + \frac{k_1}{2}, u_2 + \frac{l_1}{2}\right) = -0.01592945915,\ l_2 = hg\left(r_2 + \frac{h}{2}, R_2 + \frac{k_1}{2}, u_2 + \frac{l_1}{2}\right) = 0.03323185465$$

$$k_3 = hf\left(r_2 + \frac{h}{2}, R_2 + \frac{k_2}{2}, u_2 + \frac{l_2}{2}\right) = -0.0159824995,\ l_3 = hg\left(r_2 + \frac{h}{2}, R_2 + \frac{k_2}{2}, u_2 + \frac{l_2}{2}\right) = 0.033229580968$$

$$k_4 = hf\left(r_2 + h, R_2 + k_3, u_2 + l_3\right) = -0.014321134,\ l_4 = hg\left(r_2 + h, R_2 + k_3, u_2 + l_3\right) = 0.03224847231$$

$$R_3 = R_2 + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right) = 4.9415668798$$

Fourth iteration

$$k_1 = hf\left(r_3, R_3, u_3\right) = -0.01431969217,\ l_1 = hg\left(r_3, R_3, u_3\right) = 0.0322484765$$

$$k_2 = hf\left(r_3 + \frac{h}{2}, R_3 + \frac{k_1}{2}, u_3 + \frac{l_1}{2}\right) = -0.0127072683,\ l_2 = hg\left(r_3 + \frac{h}{2}, R_3 + \frac{k_1}{2}, u_3 + \frac{l_1}{2}\right) = 0.0313377685647$$

$$k_3 = hf\left(r_3 + \frac{h}{2}, R_3 + \frac{k_2}{2}, u_3 + \frac{l_2}{2}\right) = -0.0127528037,\ l_3 = hg\left(r_3 + \frac{h}{2}, R_3 + \frac{k_2}{2}, u_3 + \frac{l_2}{2}\right) = 0.03133590195$$

$$k_4 = hf\left(r_3 + h, R_3 + k_3, u_3 + l_3\right) = -0.0111861019759,\ l_4 = hg\left(r_3 + h, R_3 + k_3, u_3 + l_3\right) = 0.030488002387$$

$$R_4 = R_3 + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right) = 4.9288292234$$

Fifth iteration

$$k_1 = hf\left(r_4, R_4, u_4\right) = -0.01118496,\ l_1 = hg\left(r_4, R_4, u_4\right) = 0.03048799906584$$

$$k_2 = hf\left(r_4 + \frac{h}{2}, R_4 + \frac{k_1}{2}, u_4 + \frac{l_1}{2}\right) = -0.00966056,\ l_2 = hg\left(r_4 + \frac{h}{2}, R_4 + \frac{k_1}{2}, u_4 + \frac{l_1}{2}\right) = 0.02969603582$$

$$k_3 = hf\left(r_4 + \frac{h}{2}, R_4 + \frac{k_2}{2}, u_4 + \frac{l_2}{2}\right) = -0.009700160048,\ l_3 = hg\left(r_4 + \frac{h}{2}, R_4 + \frac{k_2}{2}, u_4 + \frac{l_2}{2}\right) = 0.0296944798$$

$$k_4 = hf\left(r_4 + h, R_4 + k_3, u_4 + l_3\right) = -0.00821551,\ l_4 = hg\left(r_4 + h, R_4 + k_3, u_4 + l_3\right) = 0.02895282206$$

$$R_5 = R_4 + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right) = 4.91914223682$$

Sixth iteration

$$k_1 = hf\left(r_5, R_5, u_5\right) = -0.0082145976, \ l_1 = hg\left(r_5, R_5, u_5\right) = 0.0289528142$$

$$k_2 = hf\left(r_5 + \frac{h}{2}, R_5 + \frac{k_1}{2}, u_5 + \frac{l_1}{2}\right) = -0.0067669569, \ l_2 = hg\left(r_5 + \frac{h}{2}, R_5 + \frac{k_1}{2}, u_5 + \frac{l_1}{2}\right) = 0.02825627368$$

$$k_3 = hf\left(r_5 + \frac{h}{2}, R_5 + \frac{k_2}{2}, u_5 + \frac{l_2}{2}\right) = -0.068017839, \ l_3 = hg\left(r_5 + \frac{h}{2}, R_5 + \frac{k_2}{2}, u_5 + \frac{l_2}{2}\right) = 0.0282549598$$

$$k_4 = hf\left(r_5 + h, R_5 + k_3, u_5 + l_3\right) = -0.0053891016, \ l_4 = hg\left(r_5 + h, R_5 + k_3, u_5 + l_3\right) = 0.0275992801$$

$$R_6 = R_5 + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right) = 4.91235204$$

Seventh iteration

$$k_1 = hf\left(r_6, R_6, u_6\right) = -0.0053883549, \ l_1 = hg\left(r_6, R_6, u_6\right) = 0.027599269$$

$$k_2 = hf\left(r_6 + \frac{h}{2}, R_6 + \frac{k_1}{2}, u_6 + \frac{l_1}{2}\right) = -0.00400839146, \ l_2 = hg\left(r_6 + \frac{h}{2}, R_6 + \frac{k_1}{2}, u_6 + \frac{l_1}{2}\right) = 0.026980460684$$

$$k_3 = hf\left(r_6 + \frac{h}{2}, R_6 + \frac{k_2}{2}, u_6 + \frac{l_2}{2}\right) = -0.0040393319, \ l_3 = hg\left(r_6 + \frac{h}{2}, R_6 + \frac{k_2}{2}, u_6 + \frac{l_2}{2}\right) = 0.0269793394$$

$$k_4 = hf\left(r_6 + h, R_6 + k_3, u_6 + l_3\right) = -0.002690421, \ l_4 = hg\left(r_6 + h, R_6 + k_3, u_6 + l_3\right) = 0.0263941305$$

$$R_7 = R_6 + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right) = 4.90832300287$$

Eighth iteration

$$k_1 = hf\left(r_7, R_7, u_7\right) = -0.0026898049, \ l_1 = hg\left(r_7, R_7, u_7\right) = 0.02639411856$$

$$k_2 = hf\left(r_7 + \frac{h}{2}, R_7 + \frac{k_1}{2}, u_7 + \frac{l_1}{2}\right) = -0.001370099, \ l_2 = hg\left(r_7 + \frac{h}{2}, R_7 + \frac{k_1}{2}, u_7 + \frac{l_1}{2}\right) = 0.0258393914$$

$$k_3 = hf\left(r_7 + \frac{h}{2}, R_7 + \frac{k_2}{2}, u_7 + \frac{l_2}{2}\right) = -0.001397835, \ l_3 = hg\left(r_7 + \frac{h}{2}, R_7 + \frac{k_2}{2}, u_7 + \frac{l_2}{2}\right) = 0.0258384261$$

$$k_4 = hf\left(r_7 + h, R_7 + k_3, u_7 + l_3\right) = -0.0001059623, \ l_4 = hg\left(r_7 + h, R_7 + k_3, u_7 + l_3\right) = 0.025311631$$

$$R_8 = R_7 + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right) = 4.906934396869$$

Ninth iteration

$$k_1 = hf\left(r_8, R_8, u_8\right) = -0.0001054485, \ l_1 = hg\left(r_8, R_8, u_8\right) = 0.02531161856$$

$$k_2 = hf\left(r_8 + \frac{h}{2}, R_8 + \frac{k_1}{2}, u_8 + \frac{l_1}{2}\right) = 0.0116013240, \ l_2 = hg\left(r_8 + \frac{h}{2}, R_8 + \frac{k_1}{2}, u_8 + \frac{l_1}{2}\right) = 0.02481028401$$

$$k_3 = hf\left(r_8 + \frac{h}{2}, R_8 + \frac{k_2}{2}, u_8 + \frac{l_2}{2}\right) = 0.01135065677, \ l_3 = hg\left(r_8 + \frac{h}{2}, R_8 + \frac{k_2}{2}, u_8 + \frac{l_2}{2}\right) = 0.024809447$$

$$k_4 = hf\left(r_8 + h, R_8 + k_3, u_8 + l_3\right) = 0.0023754962, \ l_4 = hg\left(r_8 + h, R_8 + k_3, u_8 + l_3\right) = 0.02433156$$

$$R_9 = R_8 + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right) = 4.908077804$$

Tenth iteration

$$k_1 = hf\left(r_9, R_9, u_9\right) = 0.0023759288,\ l_1 = hg\left(r_9, R_9, u_9\right) = 0.0253062073$$

$$k_2 = hf\left(r_9 + \frac{h}{2}, R_9 + \frac{k_1}{2}, u_9 + \frac{l_1}{2}\right) = 0.0036412392,\ l_2 = hg\left(r_9 + \frac{h}{2}, R_9 + \frac{k_1}{2}, u_9 + \frac{l_1}{2}\right) = 0.024798888$$

$$k_3 = hf\left(r_9 + \frac{h}{2}, R_9 + \frac{k_2}{2}, u_9 + \frac{l_2}{2}\right) = 0.0036158732,\ l_3 = hg\left(r_9 + \frac{h}{2}, R_9 + \frac{k_2}{2}, u_9 + \frac{l_2}{2}\right) = 0.024798139$$

$$k_4 = hf\left(r_9 + h, R_9 + k_3, u_9 + l_3\right) = 0.0485574279,\ l_4 = hg\left(r_9 + h, R_9 + k_3, u_9 + l_3\right) = 0.02431436066$$

$$R_{10} = R_9 + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right) = 4.911655084$$

Result

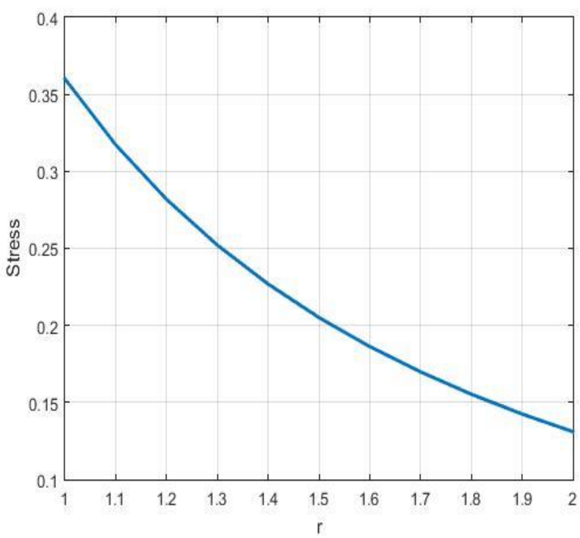

Figure 1: A Matlab plot of the current configuration vs the reference configuration

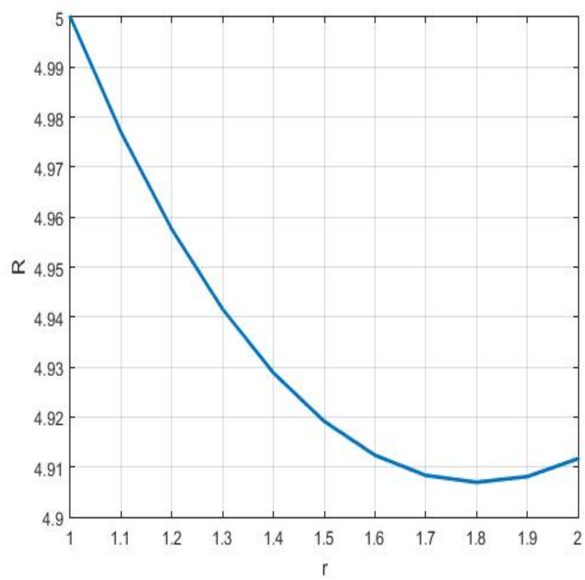

Figure 2: A Matlab plot of the stress on the material vs the reference configuration

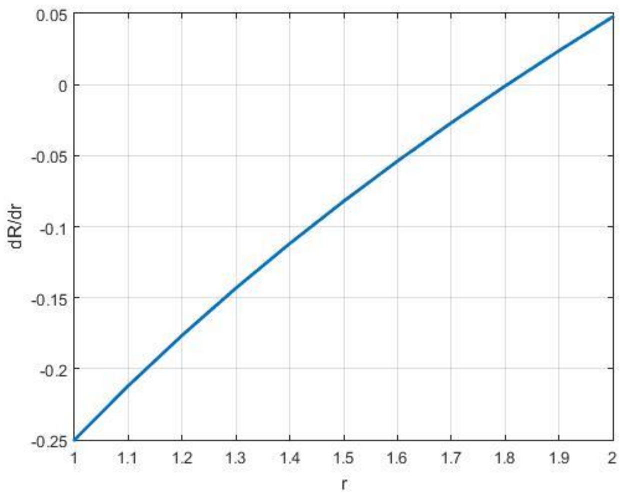

Figure 3: Effect of pressure p on the material

Discussion

From the figure 1 above, we see that the radial coordinates in the reference configuration have an inverse relationship with the radial coordinates in the current configuration. That is to say, as r increases from 1 to 2, R decreases from 5 and attains a minimum of around 4.907. This is in agreement with reality since the material under study is externally pressurized, so a decrease in volume is expected.

Figure 2 shows a plot of stress on the material, as seen in the figure, the stress decreases as we move outwards towards cylinder's circumference, so a maximum stress is observed in the interior of the cylinder and a minimum on its surface.

We have also compared the quantity dR/dr which describes the pressure on the cylinder in figure 3, we observe that the pressure increases towards the center of the cylinder. This is true because the volume of the cylinder decreases implying that its area also decreases, so it is expected that more force will be applied in order to cause a deformation.

Summary

We considered the finite deformation of a compressible cylindrical elastic material subjected to torsion and twisting effect. We started by first obtaining the Cauchy green right deformation gradient tensor and the principal strain invariants in cylindrical coordinates, then used them together with the Blatz and Ko strain energy density function for compressible material to solve for the Cauchy stresses. The obtained physical components of the Cauchy stresses were substituted into the equilibrium equations in cylindrical coordinates and a highly nonlinear second order ordinary differential equation was obtained after simplification. Due to the high nonlinearity of the model equation, no analytical solution exists. Therefore, we imposed initial and boundary conditions to the model and applied the Runge-Kutta fourth order numerical technique to solve the proposed model using a step size of 0.1 with 10 iterations. We finally computed the stresses on the material.

Conclusion

We make the following conclusions from this work:

- Compressibility plays an important role in the deformation of cylindrical elastic materials as a reduction in volume is seen for an externally applied pressure.
- Torsion and twisting have no significant effect on the radial deformation of a compressible Blatz and Ko material.
- A maximum pressure is obtained at the center of the cylinder while a minimum pressure is obtained at its circumference

Recommendations

- Other strain energy functions can be investigated using Shooting method and comparing the results with the ones presented in this work.
- Different methods can be employed to solve and be compared with the Runge-Kutta method.

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